Journal of Applied Mathematics and Mechanics 72 (2008) 347–359

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Scenarios for the quasistatic growth of a slightly curved and kinked crack☆

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ARTICLE INFO

Article history:
Received 15 March 2007

ABSTRACT

A variational-asymptotic model of the Griffith criterion for the development of a crack is constructed for a complex stress-strain state. It is assumed that the shear loads are much smaller than the breaking loads but the longitudinal loading of the crack is taken into account. Using asymptotic analysis, the problem of finding the minimum of the total energy of a body with a crack reduces to a sequence of algebraic equations, the solutions of which determine the form of the branch of the crack and its length as a function of a time-like dimensionless parameter. The absence of solutions is treated as a conversion of the fracture process to a dynamic stage and the impossibility of a quasistatic formulation of the problem. In particular, the application of shear and longitudinal loads just leads to an avalanche-type growth of the crack.

1. Formulation of the problem of the slightly curved crack

Suppose $\Omega \subset \mathbb{R}^2$ is a plane homogeneous anisotropic body with a rectilinear boundary crack L. We introduce systems of Cartesian coordinates $x = (x_1, x_2)$ and polar coordinates (r, φ) with centre O at the cut tip, and we direct the semiaxes Ox_1 and the polar axis along continuation of the cut. The equilibrium equations and the boundary conditions have the form

$$-\partial_1\sigma_{1k}(u;x) - \partial_2\sigma_{2k}(u;x) = 0, \quad x \in \Omega, \quad k = 1,2 \tag{1.1}$$

$$\sigma^{(n)}(u;x) := \sigma(u;x)n(x) = g(x), \quad x \in \partial\Omega \tag{1.2}$$

Here, $\partial_i = \partial_{xi}$, $n = (n_1, n_2)$ is the unit vector of the out ward normal to the boundary $\partial\Omega$, $\sigma_{jk}(u)$ are the Cartesian components of the stress tensor $\sigma(u)$ and $u = (u_1, u_2)$ is the displacement vector. There are no body forces and the stresses $g = (g_1, g_2)$ are applied to the external boundary of the body, that is, $g = 0$ on the surfaces $L^{\pm}$ of the crack L.

The aim of this paper is to find the trajectory of the quasistatic propagation of the branch of the crack in the case of a complex load

$$g(\tau) = g^0 + \tau g', \quad g^0 = g^1 + \varepsilon g^2 \tag{1.3}$$

Here, g^1 and g^2 are the forces which generate the stressed state of the first and second modes close to the tip respectively, τ is a dimensionless parameter and g' is the component which describes the evolution of the external stresses. The introduction of a time-like loading parameter $\tau \geq 0$ justifies us in neglecting inertial terms and ensures the formulation of the quasistatic fracture problem. The presence of the small parameter $\varepsilon > 0$ in the expansion (1.3) means that the load g^1 leads to a rectilinear propagation of the crack and the term εg^2 produces a small deviation of the branch and permits the use of asymptotic methods. The crack development trajectory has the form

$$x_2 = \varepsilon H(\varepsilon, x_1) \sim \varepsilon \sum_{m=2}^{\infty} H_m(\varepsilon) x_1^{m/2} \tag{1.4}$$

The result of the calculations which are subsequently carried out is the asymptotic of the first two coefficients $H_m(\varepsilon)$ of the series (1.4) with respect to ε and the asymptotic of the projection $h(\varepsilon, \tau)$ of the end of the branch onto the Ox_1 axis at the instant $\tau \in [0, \tau_0]$. Note that

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 3, pp. 497–515, 2008.

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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2008.07.013

the half-integer exponents x_1 are consistent with the analogous singularities of the elastic fields close to the crack tip and the form (1.4) for the branch was apparently discussed for the first time in Refs. 1 and 2.

In this paper, the trajectory of the crack is determined using the Griffith criterion in the initial formulation, since the concept of rate of energy release and the accompanying invariant integrals, which are frequently used in fracture mechanics, are insufficient in our case. In fact, following Griffith, we assume that, for any small $\tau > 0$, the total energy of a body Ω with a developing crack $L(\tau)$, which is equal to the sum of the strain potential energy and the surface energy, takes the smallest value among the energies corresponding to other possible forms and lengths of branches of the crack $\Upsilon(\tau) = L(\tau)\backslash L$. The problem of finding the optimal form is not completely solvable using present-day mathematical apparatus. Assumption (1.3) and the constraints $\varepsilon \ll 1$, $\tau \ll 1$ considerably simplify the general problem and enable us to reduce it to a recursive sequence of proper algebraic problems. In certain cases, they can become unsolvable. This means that the crack develops in an unstable and avalanche-type manner and a quasistatic formulation of the problem, which excludes dynamic effects, is inadmissible. If, however, solutions exist, then the form of the crack and the position of its tip at an instant τ are recovered using these solutions. A multiplicity of solutions is interpreted as the possibility of bifurcations. In other words, a model of the energy fracture criterion is proposed for a complex stress-strain state caused by the distortion of the crack. Similar variational-asymptotic models of different criteria have been constructed earlier[3–6] in the case of the rectilinear growth of one- and two-dimensional cracks. Note that the addition to series (1.4) of the term $x_1^{\beta/2}H_\beta(\varepsilon)$ with a non-integral exponent β introduces an additional problem into the above-mentioned recursion sequence in the solution of which we always obtain that $H_\beta = 0$. In a certain sense, this fact justifies the asymptotic formulation (1.4).

The so-called deformation basis of singular solutions of the anisotropic problem of elasticity theory in a plane with a crack is introduced in Section 2. On the whole, the properties of the elements of this basis are the same as in the case of an isotropic plane, and the sole difference is the integral characteristic (2.12), that is, the symmetric positive-definite matrix M which, in the case of an isotropic medium, is proportional to the unit matrix (see formula (5.9) below). The elements $\mathcal{M}_{p,q}^{j,k}$ of the energy release matrix for a slightly curved crack are expressed in terms of the matrix M and the terms of the series (1.4) (Sections 3 and 4). Scenarios for the quasistatic growth of a crack with and without longitudinal loading are described using the asymptotic formulae obtained in Sections 5–7. In particular, it is established when the fracture process transfers into a dynamic stage. Conditions, which ensure one or another type of crack propagation, are discussed in the concluding Section 8.

2. Singularities and weighting functions

The behaviour of the stress-strain state close to the crack tip is for the most part characterized by the two power solutions (displacement vectors) of the homogeneous theory of elasticity in a plane with a semi-infinite cut $\dot{\Lambda} = \{x : x_1 < 0, x_2 = 0\}$, which generate the root stress singularities

$$X^{j,1}(x) = r^{1/2}\Phi^{j,1}(\varphi), \quad j = 1, 2 \tag{2.1}$$

Normalizations of the basis (2.1), adapted to fracture criteria of a different physical nature, have been proposed.[7,8] It will henceforth be more convenient to use a "deformation" basis which satisfies the relations

$$[X_i^{j,1}](-r) = 8(2\pi)^{-1/2}B\delta_{3-i,j}r^{1/2}, \quad i, j = 1, 2 \tag{2.2}$$

Here, $[\upsilon](x_1) = \upsilon(x_1, +0) - \upsilon(x_1, -0)$ is the jump in the function υ on the cut surfaces, $\delta_{j,k}$ is the Kronecker delta and the factor $B = B_{11,11}$ is an element of the pliability tensor which is the inverse of the stiffness tensor A in Hooke's law $\sigma = A\varepsilon$ (ε is the strain tensor with components $\varepsilon_{jk} = (\partial_j u_k + \partial_k u_j)/2$). The power solutions, satisfying the normalization conditions (2.2), possess additional properties[8] which simplify the asymptotic analysis:

$$X_1^{1,1}(x_1, \pm 0) = 0, \quad X_1^{2,1}(x_1, +0) = -X_1^{2,1}(x_1, -0), \quad x_1 < 0 \tag{2.3}$$

$$\sigma_{11}(U^j; x_1, \pm 0) = B^{-1}\partial_1 X_1^{j,1}(x_1, \pm 0) = \mp 2\delta_{j,2}(2\pi r)^{-1/2}, \quad x_1 < 0, \quad j = 1, 2 \tag{2.4}$$

$$\partial_2 X^{1,1} = -\partial_1 X^{2,1} \tag{2.5}$$

In the case of an isotropic material, the deformation and usual force bases are identical and relations (2.3)–(2.5) are a direct corollary of the explicit formulae.

According to general results[9] (see also the Refs. 10 and 7) a basis of "non-energy" power solutions with a homogeneity exponent of $-1/2$ exists

$$Y^{p,1}(x) = r^{-1/2}\Psi^{p,1}(\varphi), \quad p = 1, 2 \tag{2.6}$$

which is subject to the biorthogonality conditions

$$Q(X^{j,k}, Y^{p,q}; \Gamma) = \delta_{j,k}\delta_{p,q}, \quad j, p = 1, 2 \tag{2.7}$$

While $k = q = 1$ in relations (2.7), we shall later extend them to other values of the indices. The antisymmetric form

$$Q(u, \upsilon; \Gamma) = \int_\Gamma \{\upsilon(x)\cdot\sigma^{(n)}(u; x) - u(x)\cdot\sigma^{(n)}(\upsilon; x)\}ds \tag{2.8}$$

$$Q(u, \upsilon; \Gamma) = -Q(\upsilon, u; \Gamma) \tag{2.9}$$

as a contour integral in Green's formula. Here, ds is an element of length of a simple smooth arc Γ, joining the surfaces of the cut Λ and the containing the tip O, and n is the unit outward normal to the boundary of the domain which is contained within Γ. If the fields u and v satisfy the homogeneous equilibrium equations and the homogeneous boundary conditions (1.2), then the integral (2.8) is invariant. The relation, which resembles the rule for integration by parts,

$$Q(\partial_1 u, v; \Gamma) = -Q(u, \partial_1 v; \Gamma) \tag{2.10}$$

has been proved.[11]

The derivatives $\partial_1 X^{j,1}$ of solutions (2.1) along the crack remain power solutions but acquire an exponent $-1/2$ and are therefore decomposed using the basis (2.6)

$$\partial_1 X^{j,1}(x) = -M_{j1}Y^{1,1}(x) - M_{j2}Y^{2,1}(x), \quad j = 1, 2 \tag{2.11}$$

The equalities

$$M_{jk} = Q(\partial_1 X^{j,1}, X^{k,1}; \Gamma) = Q(\partial_1 X^{k,1} X^{j,1}; \Gamma) = M_{kj} \tag{2.12}$$

hold by virtue of formulae (2.7) and (2.10). Consequently, the 2×2 matrix $M = (M_{jk})$ is symmetric. It has been verified[11,7] that it is positive-definite.

The derivative $\partial_2 X^{2,1}$ is not a power solution on account of the fact that the right-hand side of formula (2.4) is non zero when $j=2$. By virtue of equality (2.5), the derivative $\partial_2 X^{1,1}$ is a power solution and, according to representations (2.5) and (2.11), we have

$$\partial_2 X^{1,1}(x) = M_{12}Y^{1,1}(x) + M_{22}Y^{2,1}(x) \tag{2.13}$$

A further group of power solutions

$$X^{j,3}(x) = r^{3/2}\Phi^{j,3}(\varphi), \quad Y^{j,3}(x) = r^{-3/2}\Psi^{j,3}(\varphi), \quad j = 1, 2 \tag{2.14}$$

satisfies the relations[10]

$$X^{j,1}(x) = 2\partial_1 X^{j,3}(x), \quad Y^{j,3}(x) = -\frac{1}{2}\partial_1 Y^{j,1}(x), \quad j = 1, 2 \tag{2.15}$$

It follows from this and from formula (2.10) that equalities (2.7) hold when $k,q = 1, 3$.

Finally, there are power solutions

$$X^{1,2}(x) = r^1\Phi^{1,2}(\varphi), \quad Y^{1,2}(x) = r^{-1}\Psi^{1,2}(\varphi) \tag{2.16}$$

The first of them depends linearly on the variable x and corresponds to stretching along the crack

$$\sigma_{11}(X^{1,2}; x) = 1, \quad \sigma_{12}(X^{1,2}; x) = \sigma_{22}(X^{1,2}; x) = 0 \tag{2.17}$$

The second is normalized by conditions (2.7), which are now extended to all values of the indices k, $q = 1, 2, 3$. The solutions $X^{2,2}$ and $Y^{2,2}$, which are analogous to solutions (2.16), correspond to rotation about the point O and to a moment concentrated at this point, but do not occur in the subsequent calculations. Note that equalities (2.17) determine the component $\partial_1 X_1^{1,2} = B\sigma_{1i}(X^{1,2}) = B$ of the strain tensor, and this means that $X_1^{1,2}(x) = Bx_1 + Cx_2$ and

$$X_1^{1,2}(x_1, \pm 0) = Bx_1 \tag{2.18}$$

The above mentioned power solutions form the asymptotic of the solution $u^0(x)$ of the problem of a crack in a body Ω under a load g^0

$$u^0(x) = c(x) + \sum_{j=1}^{2}(K_{j,1}^0 X^{j,1}(x) + K_{j,3}^0 X^{j,3}(x)) + K_{1,2}^0 X^{1,2}(x) + O(r^2), \quad r \to +0 \tag{2.19}$$

Here, $c(x)$ is a rigid displacement, $K_{1,1}^0$ and $K_{2,1}^0$ are the stress intensity factors (SIFs) of the two modes, according to equalities (2.17), $K_{1,2}^0$ is the bounded part of the longitudinal stress at the point O, and $K_{1,3}^0$ and $K_{2,3}^0$ are the coefficients of the lowest stress singularities ("lowest SIFs"). We will denote the SIFs for the solution $u'(x)$ of the same problem, but in the case of a load g', by $K_{j,1}'$ etc.

Since the normalization conditions (2.2) operate with the jumps in the displacements on the crack surfaces, the products $8BK_{1,1}^0$ and $8BK_{2,1}^0$ would be correctly called *strain intensity factors* (StrainIFs). The stresses in the branch of the crack are calculated[8] using the formulae

$$\sigma_{3-j2}(u^0; r, 0) = (2\pi r)^{-1/2}(2B)^{-1}(M_{j1}K_{1,1}^0 + M_{j2}K_{2,1}^0) + O(r) \; (j = 1, 2) \tag{2.20}$$

In the case of an isotropic material, $(2B)^{-1}M$ is the unit matrix, i.e., $K_{j,1}^0$ is an SIF in a classical sense. Relations (2.20) enable us to reformulate the results of Sections 5–7 in terms of classical SIFs but the intermediate calculations are simpler in the case of normalization conditions (2.2). It is therefore convenient to relate the expansions of the stress-strain state for the first and second modes with the StrainIF rather than the SIF concept for example because, in the case of isotropic strength properties of a material, the propagation of a rectilinear crack in an anisotropic body is ensured[8] by the equality $K_{2,1}^0 = 0$ in the case of StrainIFs. In order not to introduce too much new terminology, in expansions (2.16) and subsequently, we call $K^0_{j,1}$ stress intensity factors, learning in mind their conversion into classical SIFs using to relations (2.20).

The weighting functions $\xi^{j,k}$, that is, the solutions of homogeneous problem (1.1), (1.2) with the specified behaviour around the tip O

$$\zeta^{j,k}(x) = Y^{j,k}(x) + c^{j,k}(x) + \sum_{p,q} \mathcal{L}^{j,k}_{p,q} X^{p,q}(x) + O(r^2), \quad r \to +0 \tag{2.21}$$

will be subsequently required below.

They play a part in the integral representations of the SIFs (see Refs. 12,9,7, etc.)

$$K^0_{j,k} = \int_{\partial\Omega} \zeta^{j,k}(x) \cdot g^0(x) ds \tag{2.22}$$

In formulae (2.21) and (2.22), the subscripts j and k take the values indicated in definitions (2.1), (2.6) and (2.14), (2.16). The rigid displacements $c(x)$ and $c^{j,k}(x)$ do not appear in the calculations and can be chosen arbitrarily. The weighting function $\xi^{2,2}$ does not exist. The coefficients $\mathcal{L}^{j,k}_{p,q}$ in expansion (2.21) depend on the stiffness tensor A and the shape of the body Ω. The 5×5 matrix $\mathcal{L} = (\mathcal{L}^{j,k}_{p,q})$ is symmetric and is positive-definite for a boundary crack L (see Refs. 10,13,14).

We will assume that the inequality $K^0_{1,1} > 0$ holds which, by virtue of the normalization conditions (2.2), means that the crack is open. The second expansion of (1.3) shows that the SIFs $K^0_{1,1}$ and $K^0_{1,3}$ and

$$K^0_{2,1} = \varepsilon \bar{K}^0_{2,1}, \quad K^0_{2,3} = \varepsilon \bar{K}^0_{2,3} \tag{2.23}$$

are found using formulae (2.22), in which the load g is replaced by its components g^1 and εg^2 respectively. Because of the choice of the parameter $\varepsilon \geq 0$, it is possible to ensure the equality $K^0_{1,1} = |\bar{K}^0_{2,1}|$ but it is sufficient to assume that the SIFs $K^0_{1,1}$ and $\bar{K}^0_{2,1}$ are quantities of the same order of magnitude. No constraints are imposed on $K^0_{1,2}$ and $K_{j,k}{}'$.

3. Increments in the surface and potential energies

By virtue of formula (1.4), an increment in the surface energy has the form

$$\begin{aligned}
\Delta\Pi &= 2\int_0^h \gamma(\operatorname{tg}[\varepsilon H'(\varepsilon, x_1)])(1 + \varepsilon^2 H'(\varepsilon, x_1)^2)^{1/2} dx_1 = \\
&= 2\int_0^h \Big[\gamma_0 + \gamma_0'\varepsilon H'(\varepsilon, x_1) + \frac{1}{2}\gamma_0''\varepsilon^2 H'(\varepsilon, x_1)^2 + O(\varepsilon^3)\Big]\Big(1 + \frac{1}{2}\varepsilon^2 H'(\varepsilon, x_1)^2 + O(\varepsilon^4)\Big) dx_1 = \\
&= 2\gamma_0 h + 2\varepsilon\gamma_0' h[H_2 + h^{1/2}H_3 + hH_4 + O(h^{3/2})] + \\
&+ \varepsilon^2 h(\gamma_0 + \gamma_0'')\Big[H_2^2 + 2h^{1/2}H_2H_3 + h\Big(\frac{9}{8}H_3^2 + 2H_2H_4\Big) + O(h^{3/2})\Big] + O(\varepsilon^3 h) \\
&\gamma_0 = \gamma(0), \quad \gamma_0' = \gamma'(0), \quad \gamma_0'' = \gamma''(0), \quad H_m = H_m(\varepsilon), \quad m = 2, 3, 4
\end{aligned} \tag{3.1}$$

It follows that a branch of a crack $\Upsilon(\tau)$ is interpreted as a singular perturbation of the boundary of the body $\Omega\backslash L$. Procedures exist[15–17] for calculating of the asymptotic of energy functionals for general self-adjoint boundary value problems, based on the methods of composite and matched asymptotic expansions. These procedures have been adapted[10,18,14,8] for singularly perturbed problems in the theory of elasticity and, in particular, to problems of crack mechanics. An increment in the strain potential energy accompanying the formation of a distorted branch under a constant load is approximated to any accuracy by a quadratic form of the coefficients of the expansion of the solution $u^0(x)$ close to the crack tip. The quadratic form is constructed using two infinite matrices but, since the subsequent calculations are carried out with an error $O(h^{5/2})$, the 5×5 matrices $\mathcal{L}$ of coefficients of the expansions (2.21) of the singular solutions of the first limiting problem (1.1), (1.2) in the domain Ω and the matrices $\mathcal{M}$, composed of the coefficients $\mathcal{M}^{j,k}_{p,q}$ of the expansions of the solutions $w^{j,k}$ of the second limiting problem in a plane with a distorted semi-infinite cut

$$\mathcal{L} = \Lambda \cup \{\xi = (\xi_1, \xi_2) : \xi_1 \in [0, 1), \xi_2 = \varepsilon H(\varepsilon, h\xi_1) =: \varepsilon\mathcal{H}(\xi_1)\} \tag{3.2}$$

are sufficient for constricting of the quadratic form.

The second limiting problem is obtained as a result of changing to the stretched coordinates $\xi = h^{-1}x$ and its solutions $w^{j,k}$ satisfy the homogeneous equilibrium equations (1.1) in $\mathbb{R}^2\backslash\mathcal{L}$ with the homogeneous boundary conditions (1.2) on the surfaces $\mathcal{L}^{\pm}$ of the cut (3.2) and the asymptotic conditions at infinity

$$w^{j,k}(\xi) = X^{j,k}(\xi) + \sum_p \sum_q \mathcal{M}^{j,k}_{p,q} Y^{p,q}(\xi) + \mathcal{M}^{j,k}_{1,2} Y^{1,2}(\xi) + O(|\xi|^{-5/2}), \quad |\xi| \to \infty \tag{3.3}$$

A modification[17] of the method of matched asymptotic expansions gives[14] the following matrix representation for the difference ΔU in the strain potential energy of the body $\Omega\backslash L$ under a load g^0 and of the body $\Omega\backslash L(\tau)$ under a load $g = g + \tau g'$

$$\Delta U = U^{\tau} - \frac{1}{2}(\mathcal{K}^0 + \tau\mathcal{K}')^{\top}\mathcal{E}(\mathcal{M}^{-1} - \mathcal{E}\mathcal{L}\mathcal{E})^{-1}\mathcal{E}(\mathcal{K}^0 + \tau\mathcal{K}') + O(h^{5/2}) \tag{3.4}$$

$$U^{\tau} = -\frac{1}{2}\tau\int\limits_{\partial\Omega}(g^0\cdot u' + g'\cdot u^0)ds + O(\tau^2) \tag{3.5}$$

$$\mathcal{K} = (K^0_{1,1}, K^0_{2,1}, K^0_{1,2}, K^0_{1,3}, K^0_{2,3})^{\top}, \quad \mathcal{E} = \mathrm{diag}\{h^{1/2}, h^{1/2}, h^1, h^{3/2}, h^{3/2}\} \tag{3.6}$$

Here $\top$ is the sign of transposition, the magnitude of (3.5) is independent of the form of the branch and, in particular, of the parameter h, and K′ is a column, similar to (3.6), of SIFs generated by a load g'. Applying Neumann's formula to the matrix $(\mathcal{M}^{-1} - \mathcal{E}\mathcal{L}\mathcal{G})^{-1}$, taking account of the smallness of the diagonal matrix $\mathcal{G}$ and discarding the terms $o(h^2)$, we transform expression (3.4) to the form

$$\begin{aligned}\Delta U = U^{\tau} - \frac{h}{2}\sum_j(K^0_{j,1} + \tau K'_{j,1})\Big\{\sum_p \mathcal{M}^{j,1}_{p,1}(K^0_{p,1} + \tau K'_{p,1}) + 2h^{1/2}\mathcal{M}^{j,1}_{1,2}(K^0_{1,2} + \tau K'_{1,2}) + \\ + h\sum_p\sum_q\sum_m \mathcal{M}^{j,1}_{p,1}\mathcal{L}^{p,1}_{q,1}\mathcal{M}^{q,1}_{m,1}(K^0_{m,1} + \tau K'_{m,1}) + 2h\sum_p \mathcal{M}^{j,1}_{p,3}(K^0_{p,3} + \tau K^{\mathcal{M}}_{p,3})\Big\} - \\ - \frac{h^2}{2}\mathcal{M}^{1,2}_{1,2}(K^0_{1,2} + \tau K'_{1,2})^2 + O(h^{5/2})\end{aligned} \tag{3.7}$$

Here and henceforth, summation is carried out over the indices $j, p, q, m = 1, 2$. Formula (3.7) can be obtained directly using asymptotic methods without the intermediate expressions (3.4) (see Refs. 15,16,10,18,8). However, matrix notation[17] simplifies the calculations considerably.

The matrix $\mathcal{M}$ with the elements $\mathcal{M}^{j,k}_{p,q} = \mathcal{M}^{j,k}_{p,q}(h, H)$ is a 5×5 matrix since the indices $j = k = 2$ and $p = q = 2$ are not included in formulae (3.3) and (3.7) and it is called the (truncated) energy release matrix (for details, see Ref. 14 and previous publications[15,10,17]). The matrix $\mathcal{M}$ is symmetric and positive-definite. If the branch $\Upsilon(\tau)$ lies on the continuation of the crack and, correspondingly, $H = 0$ in formula (1.4), then $\mathcal{M}(h, 0)$ is found explicitly[14] and, in particular, the 2×2-block $(\mathcal{M}^{j,1}_{p,1}(h, 0))^2_{j,p=1}$ is proportional[7,19] to the matrix M of the coefficients of the representation (2.11). In order to calculate $\mathcal{M}(h, H)$ in the case of a curved crack, the solutions $w^{j,k}$ of the model problem in $\mathbb{R}^2\backslash L$ have to be completely determined. However, in view of the smallness of the angle of deviation of the branch, only a few terms of the asymptotic

$$\mathcal{M}^{j,k}_{p,q} = \mathcal{M}^{j,k(0)}_{p,q} + \varepsilon^1\mathcal{M}^{j,k(1)}_{p,q} + \varepsilon^2\mathcal{M}^{j,k(2)}_{p,q} + O(\varepsilon^3) \tag{3.8}$$

are required for the purposes of this paper.

According to the notation (3.2) and relation (1.4), we have

$$\mathcal{H}(\xi_1) = \mathcal{H}_2\xi_1 + \mathcal{H}_3\xi_1^{3/2} + \mathcal{H}_4\xi_1^2 + \ldots, \quad \mathcal{H}_m = h^{-1+m/2}H_m \tag{3.9}$$

and, in constructing the asymptotic (3.8), we therefore make the coordinate substitution

$$\xi \mapsto \eta = (\eta_1, \eta_2) = (\xi_1 - 1, \xi_2 - \varepsilon\mathcal{H}(1)) \tag{3.10}$$

in the model problem and transfer the data from the cut (3.2) onto the ray $\Lambda_\eta = \{\eta : \eta_1 < 0, \eta_2 = 0\}$.

The method of linearization has been used previously in the case of slightly curved smooth and kinked cracks[20–22,14,8] etc. (also, see Refs. 23 and 2 for an alternative approach). A rigorous justification of this method is available (Ref. 16, Ch. 5). Examples[24,25] show that special attention has to be paid to the behaviour of the solutions close to the vertices of the distorted angles, However, no difficulties arise in the situation considered here.

4. Calculation of the energy release matrix

If $F(\xi)=|\xi|^{\lambda}F(|\xi|^{-1}\xi)$ is a homogeneous function of degree λ then, when account is taken of the change of coordinates (3.10) and an expansion in a Maclaurin series in the variable η is used, we obtain

$$\begin{aligned}
F(\xi) &= F(\eta_1+1,\eta_2+\varepsilon\mathscr{H}(1)) = |\eta|^{\lambda}F(|\eta|^{-1}\eta_1+|\eta|^{-1},|\eta|^{-1}\eta_2+|\eta|^{-1}\varepsilon\mathscr{H}(1)) = \\
&= |\eta|^{\lambda}\Big\{F(|\eta|^{-1}\eta)+|\eta|^{-1}\partial_1F(|\eta|^{-1}\eta)+|\eta|^{-1}\varepsilon\mathscr{H}(1)\partial_2F(|\eta|^{-1}\eta)+ \\
&+\frac{1}{2}|\eta|^{-2}(\partial_1^2F(|\eta|^{-1}\eta)+2\varepsilon\mathscr{H}(1)\partial_1\partial_2F(|\eta|^{-1}\eta)+\varepsilon^2\mathscr{H}(1)^2\partial_2^2F(|\eta|^{-1}\eta))+O(|\eta|^{-3})\Big\} = \\
&= F(\eta)+\partial_1F(\eta)+\varepsilon\mathscr{H}(1)\partial_2F(\eta)+\frac{1}{2}(\partial_1^2F(\eta)+2\varepsilon\mathscr{H}(1)\partial_1\partial_2F(\eta)+\varepsilon^2\mathscr{H}(1)^2\partial_2^2F(\eta))+ \\
&+O(|\eta|^{\lambda-3})
\end{aligned}\tag{4.1}$$

Henceforth, $\partial_i=\partial/\partial\eta_i$ in this section. By increasing the length of the partial sum of the series, formula (4.1) can be written with any specified accuracy $O(|\eta|^{\lambda-N})$.

We will represent the solutions of the second limiting problem mentioned in Section 3 in the form

$$w^{j,k}(\xi) = W^{j,k}(\eta) = W^{j,k(0)}(\eta)+\varepsilon W^{j,k(1)}(\eta)+\varepsilon^2W^{j,k(2)}(\eta)+\dots \tag{4.2}$$

We will initially deal with the case when $k=1$ and note that formula (4.1) transforms the asymptotic conditions in the following manner

$$\begin{aligned}
W^{j,1}(\eta) &= X^{j,1}(\eta)+\partial_1X^{j,1}(\eta)+\varepsilon\mathscr{H}(1)\partial_2X^{j,1}(\eta)+ \\
&+\frac{1}{2}\{\partial_1^2X^{j,1}(\eta)+2\varepsilon\mathscr{H}(1)\partial_1\partial_2X^{j,1}(\eta)+\varepsilon^2\mathscr{H}(1)^2\partial_2^2X^{j,1}(\eta)\}+ \\
&+\sum_{p=1}^{2}\{\mathscr{M}_{p,1}^{j,1}Y^{p,1}(\eta)+\mathscr{M}_{p,1}^{j,1}\partial_1Y^{p,1}(\eta)+\varepsilon\mathscr{H}(1)\mathscr{M}_{p,1}^{j,1}\partial_2Y^{p,1}(\eta)+\mathscr{M}_{p,3}^{j,1}Y^{p,3}(\eta)\}+ \\
&+\mathscr{M}_{1,2}^{j,1}Y^{1,2}(\eta)+O(\rho^{-2})
\end{aligned}\tag{4.3}$$

We now consider the boundary conditions when $\zeta_1<0$ or, what is the same by virtue of relation (3.10), when $\eta_1<-1$:

$$\begin{aligned}
\sigma_{2k}(W^{j1};\eta)\big|_{\xi_2=\pm0} &= \sigma_{2k}(W^{j,1};\eta_1,-\varepsilon\mathscr{H}(1)\pm0) = \sigma_{2k}(W^{j,1};\eta_1,\pm0)- \\
&-\varepsilon\mathscr{H}(1)\partial_2\sigma_{2k}(W^{j,1};\eta_1,\pm0)+\frac{1}{2}\varepsilon^2\mathscr{H}(1)^2\partial_2^2\sigma_{2k}(W^{j,1};\eta_1,\pm0)+O(\varepsilon^{5/2})
\end{aligned}\tag{4.4}$$

Since, in the distorted parts of the surfaces $\mathscr{L}^{\pm}$, the normal (which is not a unit normal) has the form $(\pm\varepsilon\mathscr{H}'(\xi_1),\mp1)$ and the equality $\eta_2=\varepsilon(\mathscr{H}(\xi_1)-\mathscr{H}(1)\pm0$, is satisfied, by analogy with the calculation of (4.4), we find using Taylor's formula with respect to the variable ξ_2

$$\begin{aligned}
&\mp(1+\varepsilon^2\mathscr{H}'(\xi_1)^2)^{1/2}\sigma_k^{(n)}(W^{j,1};\eta)\big|_{\xi_2=\varepsilon\mathscr{H}(\xi_1)\pm0} = \sigma_{2k}(W^{j,1};\eta_1,\pm0)- \\
&-\varepsilon\{\mathscr{H}'(\xi_1)\sigma_{1k}(W^{j,1};\eta_1,\pm0)-(\mathscr{H}(\xi_1)-\mathscr{H}(1))\partial_2\sigma_{2k}(W^{j,1};\eta_1,\pm0)\}- \\
&-\varepsilon^2(\mathscr{H}(\xi_1)-\mathscr{H}(1))\Big\{\mathscr{H}'(\xi_1)\partial_2\sigma_{1k}(W^{j,1};\eta_1,\pm0)-\frac{1}{2}(\mathscr{H}(\xi_1)-\mathscr{H}(1))\partial_2^2\sigma_{2k}(W^{j,1};\eta_1,\pm0)\Big\}+ \\
&+O(\varepsilon^{5/2})
\end{aligned}\tag{4.5}$$

Note that the factor $\mathscr{H}(\xi_1)-\mathscr{H}(1)$ vanishes at the point $\eta=0$ and compensates for the singularities existing in the derivatives of the stresses. Since the derivative $\mathscr{H}'(-1)$, generally speaking, is non-zero, the factors accompanying ε and ε^2 on the right-hand sides of relations (4.4) and (4.5) can acquire singularities of the first kind at the point $\eta=(0,-1)$, the image of the vertices of the straightened out angles. Discontinuities actually arise in these problems for $W^{2,1(1)}$ and $W^{1,1(2)}$ and it is well known (Refs. 24,25 and 16, Ch.5) that constructions of an additional boundary layer are required when determining the next terms $W^{2,1(2)}$. By a fortunate coincidence of circumstances, the terms $W^{2,1(2)}$ and $W^{1,1(3)}$ will not be required later.

We substitute formulations (4.2) and (3.9) into formulae (4.3)–(4.5) and collect the coefficients of like powers of the parameter ε. Each term of expansion (4.2) satisfies the homogeneous equilibrium equations in $\mathbb{R}^2\backslash\bar{\Lambda}_{\eta}$. Using these equations and the boundary conditions obtained in formulae (4.4) and (4.5), we find the relations which the terms $W^{j,1(m)}$ satisfy. In the simplest case when $m=0$, we have

$$\sigma_{2k}(W^{j,1(0)};\eta_1,\pm0) = 0, \quad \eta_1<0, \quad k = 1,2 \tag{4.6}$$

$$W^{j,1(0)}(\eta) = X^{j,1}(\eta) + \partial_1 X^{j,1}(\eta) + \frac{1}{2}\partial_1^2 X^{j,1}(\eta) + \mathcal{M}_{1,2}^{j,1(0)} Y^{1,2}(\eta) +$$
$$+ \sum_{p=1}^{2} \{\mathcal{M}_{p,1}^{j,1(0)} Y^{p,1}(\eta) + \mathcal{M}_{p,1}^{j,1(0)} \partial_1 Y^{p,1}(\eta) + \mathcal{M}_{p,3}^{j,1(0)} Y^{p,3}(\eta)\} + O(\rho^{-2}), \quad \rho \to +\infty \tag{4.7}$$

We will give the solution $W^{j,1(0)}$ explicitly. Actually, the vector function $W^{j,1(0)} = X^{j,1}$ satisfies equilibrium equations (1.1) in $\mathbb{R}^2 \backslash \bar{\Lambda}_\eta$, boundary conditions (4.6) and growth conditions (4.7). Hence, all the terms on the right-hand side of relation (4.7), with the exception of $X^{j,1}(\eta)$, mutually cancel each other out. From this, using relations (2.11) and (2.15) for the singular solutions, we derive the equalities

$$\mathcal{M}_{p,1}^{j,1(0)} = M_{jp}, \quad \mathcal{M}_{p,3}^{j,1(0)} = M_{jp}, \quad \mathcal{M}_{1,2}^{j,1(0)} = 0 \tag{4.8}$$

Relations (4.4) and (4.5) give the boundary conditions on the surfaces $\Lambda_\eta^\pm$ of the cut Λ_η for the second terms $W^{j,1(1)}$ of formulation (4.2)

$$\sigma_{2k}(W^{j,1(1)}) = \mathcal{H}(1)\partial_2\sigma_{2k}(X^{j,1}), \quad \eta_1 \in (-\infty, -1), \quad \eta_2 = \pm 0$$
$$\sigma_{2k}(W^{j,1(1)}) = \mathcal{H}'\sigma_{1k}(X^{j,1}) - (\mathcal{H} - \mathcal{H}(1))\partial_2\sigma_{2k}(X^{j,1}), \quad \eta_1 \in (-1, 0), \quad \eta_2 = \pm 0 \tag{4.9}$$

Taking account of the equilibrium equation

$$\partial_2\sigma_{2k}(X^{j,1}) = -\partial_1\sigma_{1k}(X^{j,1})$$

and property (2.4) of the deformation basis, we convert boundary conditions (4.9) to the form

$$\sigma_{2k}(W^{1,1(1)}; \eta_1 \pm 0) = 0, \quad k = 1, 2, \quad \sigma_{22}(W^{2,1(1)}; \eta_1, \pm 0) = 0, \quad \eta_1 < 0 \tag{4.10}$$

$$\sigma_{12}(W^{2,1(1)}; \eta_1, \pm 0) = -\mathcal{H}(1)\partial_1\sigma_{11}(X^{2,1}; \eta_1, \pm 0) = \pm\mathcal{H}(1)(2\pi)^{-1/2}\rho^{-3/2}$$
$$\eta_1 \in (-\infty, -1) \tag{4.11}$$

$$\sigma_{12}(W^{2,1(1)}; \eta_1, \pm 0) = -\partial_1(\mathcal{H}(1+\eta_1) - \mathcal{H}(1))\sigma_{11}(X^{2,1}; \eta_1, \pm 0) =$$
$$= \pm 2\partial_\rho\{(\mathcal{H}(1-\rho) - \mathcal{H}(1))(2\pi\rho)^{-1/2}\}, \quad \eta_1 \in (-1, 0) \tag{4.12}$$

Finally, separating terms of the order of ε in the asymptotic conditions (4.3), we obtain

$$W^{j,1(1)}(\eta) = \mathcal{H}(1)\partial_2 X^{j,1}(\eta) + \sum_{p=1}^{2} \mathcal{M}_{p,1}^{j,1(1)} Y^{p,1}(\eta) + \mathcal{M}_{1,2}^{j,1(1)} Y^{1,2}(\eta) + O(\rho^{-3/2}),$$
$$\rho \to +\infty \tag{4.13}$$

Note that, in expansion (4.3), the term $\mathcal{H}(1)\partial_1\partial_2 X^{j,1}(\eta)$, occurring in formula (4.3), is attached to the residue $O(\rho^{-3/2})$.

The vector function $W^{1,1(1)}$ satisfies the homogeneous boundary conditions (4.10) and, according to relation (4.13), vanishes at infinity as $O(\rho^{-1/2})$. Consequently, $W^{1,1(1)} = 0$ in view of the uniqueness of the energy solution. By virtue of equality (2.13), the right-hand side of relation (4.13) is equal to zero only in the case when

$$\mathcal{M}_{1,1}^{1,1(1)} = -\mathcal{H}(1)M_{21}, \quad \mathcal{M}_{2,1}^{1,1(1)} = \mathcal{M}_{1,1}^{2,1(1)} = -\mathcal{H}(1)M_{22}, \quad \mathcal{M}_{1,2}^{1,1(1)} = 0 \tag{4.14}$$

The factor $\mathcal{H}(1)$ is defined by formula (3.9) and depends on the small parameter h.

Boundary conditions (4.11) and (4.12) for the vector function $W^{2,1(1)}$ are non-trivial and we calculate the quantities $\mathcal{M}_{2,1}^{2,1(1)}$ and $\mathcal{M}_{1,2}^{2,1(1)}$ using Green's formula in a circle with a cut. Denoting a circle with radius $R > 1$ by Γ_R, we have

$$Q(W^{2,1(1)}, X^{2,1}; \Gamma_R) = \sum_{\pm} \pm \int_{-R}^{0} X_1^{2,1}(\eta_1, \pm 0)\sigma_{12}(W^{2,1(1)}; \eta_1, \pm 0)d\eta_1 \tag{4.15}$$

We substitute expressions (4.11) and (4.12) into the right-hand side of the equality (4.15) and it vanishes since the integrals along the upper and lower surfaces of the cut are identical according to property (2.3) of the deformation basis. Now, taking the limit as $R \to +\infty$ and taking account of formulae (2.7) and (2.9), we transform equality (4.15) to the form

$$\mathcal{M}_{2,1}^{2,1(1)} = \mathcal{H}(1)Q(\partial_2 X^{2,1}, X^{2,1}; \Gamma_1) \tag{4.16}$$

The integral $Q(\partial_2 X^{2,1}, X^{2,1}; \Gamma_R)$ is not invariant but it is independent of the radius R of the circle due to the fact that the integrand is equal to $o(\rho^{-1})$. If the crack lies in the plane of elastic symmetry, then $\mathcal{M}_{2,1}^{2,1(1)} = 0$.

Replacing the power solution $X^{2,1}$ by the solution $X^{1,2}$ in formula (4.15), we find

$$Q(W^{2,1(1)}, X^{1,2}; \Gamma_R) = \sum_{\pm} \pm \int_{-R}^{0} X_1^{1,2}(\eta_1, \pm 0)\sigma_{12}(W^{2,1(1)}; \eta_1, \pm 0)d\eta_1 \tag{4.17}$$

The right-hand side of the analogous formula

$$Q(\mathcal{H}(1)\partial_2 X^{2,1}, X^{1,2}; \Gamma_R) = \mathcal{H}(1)\sum_{\pm} \pm \int_{-R}^{0} X_1^{1,2}(\eta_1, \pm 0)\sigma_{12}(\partial_2 X^{2,1}; \eta_1, \pm 0)d\eta_1 =$$

$$= \mathcal{H}(1)\sum_{\pm} \mp \int_{-R}^{0} X_1^{1,2}(\eta_1, \pm 0)\partial_1\sigma_{11}(X^{2,1}; \eta_1, \pm 0)d\eta_1$$

is calculated using relations (2.4) and (2.18). Now, due to the normalization conditions (2.7), we have

$$\mathcal{M}_{1,2}^{2,1(1)} = -\lim_{R\to\infty} Q(W^{2,1(1)} - \mathcal{H}(1)\partial_2 X^{2,1}, X^{1,2}; \Gamma_R) =$$

$$= -\sum_{\pm} \pm (2\pi)^{-1/2} B \int_{-1}^{0} \pm\{2\partial_\rho(\mathcal{H}(1-\rho) - \mathcal{H}(1))\rho^{-1/2} - \mathcal{H}(1)\rho^{-3/2}\}\eta_1 d\eta_1 =$$

$$= 4(2\pi)^{-1/2} B \int_{0}^{1} \mathcal{H}(1-\rho)\rho^{-1/2} d\rho \tag{4.18}$$

Formulae (4.8), (4.14), (4.16) and (4.18) show several terms of the asymptotic (3.8) of the elements of the energy release matrix. We will now verify the additional equalities

$$\mathcal{M}_{1,1}^{1,1(2)} = 0, \quad \mathcal{M}_{2,1}^{1,1(2)} = 0 \tag{4.19}$$

$$\mathcal{M}_{1,2}^{1,2(0)} = 0, \quad \mathcal{M}_{1,2}^{1,2(1)} = 0 \tag{4.20}$$

We first find the boundary conditions for the third term $W^{1,1(2)}$ of formulation (4.2) which, according to representations (4.3) and (3.9), satisfies the conditions

$$W^{1,1(2)}(\eta) = \mathcal{M}_{1,1}^{1,1(2)} Y^{1,1}(\eta) + \mathcal{M}_{2,1}^{1,1(2)} Y^{2,1}(\eta) + \mathcal{M}_{1,2}^{1,1(2)} Y^{1,2}(\eta) + O(\rho^{-3/2}), \quad \rho \to +\infty$$

Since $W^{1,1(0)} = X^{1,1}$ and $W^{1,1(1)} = 0$, from formulae (4.4), (4.5) and (1.1), (2.5) we derive the relations

$$\sigma_{2k}(W^{1,1(2)}; \eta_1, \pm 0) = -\frac{1}{2}\mathcal{H}(1)^2\left(\frac{\partial^2}{\partial\eta_2^2}\sigma_{2k}(X^{1,1}; \eta_1, \pm 0) + \frac{\partial^2}{\partial\eta_1^2}\sigma_{11}(X^{3-k,1}; \eta_1, \pm 0)\right),$$

$$\eta_1 < -1$$

$$\sigma_{2k}(W^{1,1(2)}; \eta_1, \pm 0) = \frac{1}{2}\frac{\partial}{\partial\eta_1}\left((\mathcal{H}(1+\eta_1) - \mathcal{H}(1))^2\frac{\partial}{\partial\eta_2}\sigma_{1k}(X^{1,1}; \eta_1, \pm 0)\right) =$$

$$= -\frac{1}{2}\frac{\partial}{\partial\eta_1}\left((\mathcal{H}(1+\eta_1) - \mathcal{H}(1))^2\frac{\partial}{\partial\eta_2}\sigma_{1k}(X^{3-k,1}; \eta_1, \pm 0)\right), \quad \eta_1 \in (-1, 0)$$

Due to the normalization conditions (2.2) and the properties (2.3) and (2.4) of the power solutions (2.1), the right-hand sides are calculated explicitly and, in the case when $k = 2$, they are zero and, in the case when $k = 1$, they have different signs on the surfaces. Consequently, the technique used when calculating (4.15) and (4.17) confirms equalities (4.19). Moreover, using relations (2.2), (2.4) and (2.18) in a similar manner to transformations (4.17) and (4.18), we calculate the coefficient

$$\mathcal{M}_{1,2}^{1,1(2)} = -\lim_{R\to\infty} Q(W^{1,2(2)}, X^{1,2}; \Gamma_R) =$$

$$= -\lim_{R\to\infty}\sum_{\pm} \pm \int_{-R}^{0} X_1^{1,2}(\eta_1, \pm 0)\sigma_{12}(W^{1,2(2)}; \eta_1, \pm 0)d\eta_1 =$$

$$= -2(2\pi)^{-1/2} B \int_{0}^{1} \rho^{-1/2}\frac{d}{d\rho}(\mathcal{H}(1-\rho) - \mathcal{H}(1))^2 d\rho \tag{4.21}$$

We will now verify that formulae (4.20) hold, that is, we will prove the relation $\mathcal{M}_{1,2}^{1,2} = O(\varepsilon^2)$. The Cartesian components of the first strain vector (2.16) depend linearly on the variables x_1 and x_2. Consequently, according to formula (4.1), the difference $X^{1,2}(\xi) - X^{1,2}(\eta)$ is a stiff translational displacement, which can be neglected in the asymptotic condition

$$W^{1,2}(\eta) = X^{1,2}(\eta) + \sum_{p=1}^{2} \mathcal{M}_{p,1}^{1,2} Y^{p,1}(\eta) + \mathcal{M}_{1,2}^{1,2} Y^{1,2}(\eta) + O(\rho^{-3/2})$$

It follows from this that $W^{1,2} = X^{1,2}$, and this means that the first equality of (4.20) holds.

Taking account of the transformations (4.4) and (4.5) and the homogeneity of the stress field (2.17), we find the boundary conditions for term $W^{1,2(1)}$ of formulation (4.2)

$$\sigma_{2k}(W^{1,2(1)}; \eta_1, \pm 0) = 0, \quad \eta_1 < -1, \quad \sigma_{22}(W^{1,2(1)}; \eta_1, \pm 0) = 0, \quad \eta_1 \in (-1, 0)$$

$$\sigma_{21}(W^{1,2(1)}; \eta_1, \pm 0) = \mathcal{H}'(\xi_1)\sigma_{11}(X^{1,2}; \eta_1, 0) = \mathcal{H}'(1-\rho), \quad \eta_1 \in (-1, 0) \tag{4.22}$$

The vector function $W^{1,2(1)}$ attenuates fades at infinity and the coefficient $\mathcal{M}_{1,2}^{1,2(1)}$ of the power solution $Y^{1,2}(\eta)$ in its expansion at infinity can be calculated using the method of (4.17). However, by virtue of formulae (4.22) and (2.18), the jumps in the functions $\sigma_{21}(W^{1,2(1)})$ and $X_1^{1,2}$ on the cut Λ_η are zero and, consequently, the following equalities hold

$$Q(W^{1,2(1)}, X^{1,2}; \Gamma_R) = 0, \quad \mathcal{M}_{1,2}^{1,2(1)} = 0$$

5. The total energy increment and a model of the Griffith criterion

According to formulae (3.1), (3.7) and (3.9), the increment in the total energy $\Delta T = \Delta U + \Delta \Pi$ due to the formation of a branch $\Upsilon(\tau)$ of a crack L can be represented as

$$\Delta T = T_0(\tau) + hT_2(\tau) + h^{3/2}T_3(\tau) + h^2 T_4(\tau) + \dots \tag{5.1}$$

$$T_k(\tau) = T_k^0 + \tau T_k^1 \tag{5.2}$$

The dots indicate a residue $O(h^{5/2} + \tau^2)$. The calculations carried out in the preceding section, reveal the dependence of expressions (5.2) on the small parameter ε. In particular, by virtue of formulae (3.1), (3.7) and (2.23), (3.8), (4.14), we have

$$\begin{aligned} T_2^0 &= 2\gamma_0 + 2\varepsilon H_2 \gamma_0' + \varepsilon^2 H_2^2(\gamma_0 + \gamma_0'') - \\ &- \frac{1}{2}((M_{11} - \varepsilon^2 H_2 M_{12})(K_{1,1}^0)^2 + 2\varepsilon(M_{12} - \varepsilon H_2 M_{22}) K_{1,1}^0 \bar{K}_{2,1}^0 + \varepsilon^2 M_{22}(\bar{K}_{2,1}^0)^2) + O(\varepsilon^3) \end{aligned} \tag{5.3}$$

In the case of the quasistatic growth of a crack, its length changes smoothly, that is, $h(\tau)$ is infinitesimally small when $\tau \to +0$. If the factor $T_2^0 = T_2(0)$ accompanying h is positive, then, in the case of a small $\tau > 0$, the functional (5.1) reaches a minimum at the point $h = 0$, that is, the crack is stationary. The case when $T_2^0 < 0$ corresponds to a jumpwise change in the length of the crack, since any increase in the parameter h $\ll 1$ causes a decrease in the magnitude of ΔT, and it is impossible to study the fracture process without taking dynamic effects into account. So, the equality

$$T_2^0 = 0 \tag{5.4}$$

serves as the necessary condition for the quasistatic growth of a crack.

We collect the factors accompanying the powers of ε^0 and ε^1 on the right-hand side of formula (5.3), equate their sum to zero and obtain

$$4\gamma_0 = M_{11}(K_{1,1}^0)^2 \tag{5.5}$$

$$4(\gamma_0' + M_{12}(K_{1,1}^0)^2)H_{20} = 2M_{12}K_{1,1}^0 \bar{K}_{2,1}^0, \quad H_{20} = H_2(0) \tag{5.6}$$

Relation (5.5) determines the critical value of the SIF $K_{1,1}^0$ of the first mode, but only approximately without taking account of shear loading. From formulae (5.6) and (5.5), we obtain

$$\varepsilon H_{20} = 2M_{12}(4\gamma_0 + M_{12}(K_{1,1}^0)^2)^{-1} K_{1,1}^0 K_{2,1}^0 = \frac{1}{2} M_{12}\left(\gamma_0 + \gamma_0' \frac{M_{12}}{M_{11}}\right)^{-1} K_{1,1}^0 K_{2,1}^0 \tag{5.7}$$

By virtue of relation (1.4), the quantity (5.7) is the approximate value of the angle of deviation of the branch $\Upsilon(\tau)$ from the axis of the crack. The asymptotic procedure can be continued and the following terms of the expansion of the function H_2 in a Taylor's series with respect to the variable ε can be calculated. However, in the most interesting case of an isotropic medium, formula (5.7) loses its meaning, since

$$M_{12} = 0, \quad \gamma_0' = 0 \tag{5.8}$$

The equalities (5.8) also hold if the crack lies in a plane of symmetry of the physical properties of the medium. In a material with anisotropic elastic properties and isotropic strength properties, the equality $M_{12}=0$ picks out[8] the directions of the *rectilinear development of cracks*, subject to the condition that $K^0_{2,1}=0$. We will henceforth assume that conditions (5.8) are satisfied and we thereby include an isotropic material with a shear modulus μ and a Poisson's ratio ν in the treatment for which

$$M_{11} = M_{22} = \mu^{-1}(1-\nu), \quad M_{12} = M_{21} = 0, \quad \gamma = \text{const} \tag{5.9}$$

Assumption (5.8) simplifies the expressions for the quantities T^0_2 and T^1_2 in expansion (5.1)

$$T^0_2 = 2\gamma_0 + \varepsilon^2 H^2_2(\gamma_0+\gamma''_0) - \\ -\frac{1}{2}(M_{11}(K^0_{1,1})^2 - 2\varepsilon^2 M_{22}(H_2K^0_{1,1} - \overline{K}^0_{2,1})\overline{K}^0_{2,1}) + O(\varepsilon^3) \tag{5.10}$$

$$T^1_2 = -M_{11}K^0_{1,1}K'_{1,1} + \varepsilon M_{22}(H_2K^0_{1,1} - \overline{K}^0_{2,1})K'_{2,1} + \varepsilon^2 M_{22}H_2\overline{K}^0_{2,1}K'_{1,1} + O(\varepsilon^3) \tag{5.11}$$

According to the Griffith fracture criterion, at any instant $\tau \geq 0$ the crack is located in a position which ensures a minimum of the total energy functional or of its increment (5.1). Since the quantity $T_0(\tau)=U^\tau$ is independent of a variation in the form of a crack, we replace the increment ΔT by the sum of the three asymptotic terms

$$hT_2(\tau) + h^{3/2}T_3(\tau) + h^2T_4(\tau) \tag{5.12}$$

Seeking the minimum of the functional (5.12), we approximately calculate the unknowns $h(\tau)$ and $H(\varepsilon, x_1)$, describing the length and form of the branch of the crack. Equality (5.4) determines the critical load. We specify the instant $\tau=0$ such that the load $g(0)=g^0=g^1+\varepsilon g^2$ turns out to be precisely the critical load. Expression (5.10) depends quadratically on the coefficient $H_2(\varepsilon)$ of series (1.4) and, according to the Griffith criterion, the quantity εH_2 which specifies the angle of deviation of the crack from the initial direction is such that, other conditions being equal, the first term in the trinomial (5.12) takes the least value (at this stage, the remaining terms should be neglected in view of the smallness of the parameters h and τ). Hence, by assumption (5.8), we find

$$H_2 = -\frac{1}{2}(\gamma_0+\gamma''_0)^{-1}M_{22}K^0_{1,1}\overline{K}^0_{2,1} + O(\varepsilon) \tag{5.13}$$

The Equality (5.4) shows that a crack L is in equilibrium in the case of the following condition associating the SIFs $K^0_{1,1}$ and $\bar{K}^0_{2,1}=\varepsilon^{-1}K^0_{1,1}$ with the surface energy density

$$4\gamma_0 = M_{11}(K^0_{1,1})^2 + \varepsilon^2 M_{22}(\overline{K}^0_{2,1})^2\left(1+\frac{1}{2}(\gamma_0+\gamma''_0)^{-1}M_{22}(K^0_{1,1})^2\right) + O(\varepsilon^3) \tag{5.14}$$

Without any loss of accuracy, relations (5.14) and (5.13) can be rewritten in the form

$$4\frac{\gamma(0)}{M_{11}} = (K^0_{1,1})^2 + (K^0_{2,1})^2\frac{M_{22}}{M_{11}}\left(1+2\left(1+\frac{\gamma''_0}{\gamma_0}\right)^{-1}\frac{M_{22}}{M_{11}}\right) + O(\varepsilon^3) \tag{5.15}$$

$$\varepsilon H_2 = -2\left(1+\frac{\gamma''_0}{\gamma_0}\right)^{-1}\frac{M_{22}}{M_{11}}\frac{K^0_{2,1}}{K^0_{1,1}} + O(\varepsilon^2) \tag{5.16}$$

In the case of an isotropic medium (5.9) $M_{22}/M_{11}=1$ and $\gamma_0''=0$ and, therefore,

$$4\mu(1-\nu)^{-1}\gamma_0 = (K^0_{1,1})^2 + 3(K^0_{2,1})^2 + O(\varepsilon^3), \quad \varepsilon H_2 = -2K^0_{2,1}(K^0_{1,1})^1 + O(\varepsilon^2)$$

6. Scenarios for the growth of a crack under longitudinal loading

We will assume that the coefficient $K^0_{1,2}$ in expansion (2.19) of the solution $u^0(x)$ of problem (1.1), (1.2) in non-zero in the case of a body Ω/L under a load g^0. According to relations (2.16) and (2.17), $K^0_{1,2}$ is the value of the regular component of the stress $\sigma_{11}(u^0)$ at the crack tip, $K^0_{1,2}>0$ in the case of longitudinal stretching of the crack and $K^0_{1,2}<0$ in the case of longitudinal compression.

We note that, by virtue of relations (3.9),

$$\mathcal{H}(1) = H_2(\varepsilon) + h^{1/2}H_3(\varepsilon) + O(h^2)$$

On separating out the terms of the order of $h^{3/2}$ in asymptotic representations (3.1) and (3.7) and taking account of formulae (4.14) and (4.18), (4.21), we obtain that the factor $T^0_3=T_3(0)$ in the trinomial (5.12) has the form

$$T^0_3 = 2\varepsilon^2 H_2H_3(\gamma_0+\gamma''_0) + \varepsilon^2 H_2M_{22}K^0_{1,1}\overline{K}^0_{2,1} - \\ -\frac{8}{3}(2\pi)^{-1/2}\varepsilon^2 H_2B(H_2K^0_{1,1} - 2\overline{K}^0_{2,1})K^0_{1,2} + O(\varepsilon^3) \tag{6.1}$$

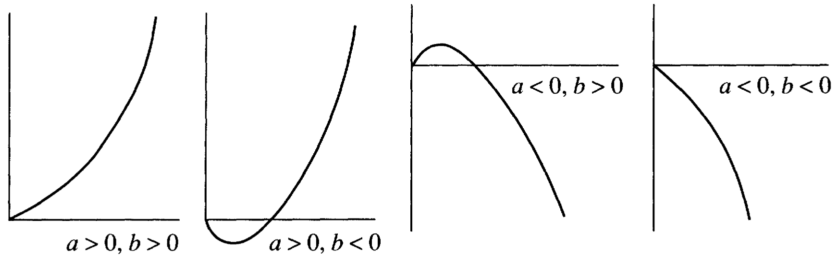

Fig. 1.

The first two terms on the right-hand side of relation (6.1) mutually cancel owing to formula (5.13) for H_2. This fact, which is important for the successive application of the Griffith criterion, eliminates the quantity H_3 from the expression for T_3^0 and, in the case of small τ and h, reduces the problem of finding the minimum of the functional (5.12) to an investigation of the function

$$\mathbb{R}_+ \ni h \mapsto ah^{3/2} + bh$$

A graph of this function is shown schematically in Fig. 1 for different constraints imposed on the coefficients

$$a = \frac{32}{3}(2\pi)^{-1/2}\varepsilon^2 B\left(1+\frac{\gamma_0''}{\gamma_0}\right)^{-2}\frac{M_{22}}{M_{11}}\left(1+\frac{\gamma_0''}{\gamma_0}+\frac{M_{22}}{M_{11}}\right)\frac{(\overline{K}_{2,1}^0)^2}{K_{1,1}^0}K_{1,2}^0 + O(\varepsilon^3),\quad b = \tau T_2^1 \tag{6.2}$$

Hence, one of the following possibilities is realized when $K_{1,2}^0 \neq 0$.

1°. If $K_{1,2}^0 > 0$ and $T_2^1 > 0$ (the stretching along the crack and the SIF decrease), the crack is stationary, that is, a minimum of the functional (5.12) is obtained when $h = 0$.
2°. If $K_{1,2}^0 >$ and $T_2^1 < 0$ (the stretching along the crack and the SIF increase), the crack develops quasistatically and stably and the position of its tip at an instant τ is determined using the formula

$$h(\tau) = \left(\frac{2}{3}\frac{T_2^1}{T_3^0}\right)^2\tau^2 + O(\tau^3) \tag{6.3}$$

3°. If $K_{1,2}^0 < 0$ and $T_2^1 < 0$ (compression along the crack and the SIF increase), the position of the crack is unstable: it can remain stationary but, by virtue of perturbations due to asymptotic terms which have not been taken into account, it can transfer to the position (6.3), corresponding to a maximum of the functional (5.12). Further lengthening of the crack causes an avalanche-type growth.
4°. If $K_{1,2}^0 < 0$ and $T_2^1 > 0$ (compression along the crack and the SIF decrease), the crack grows in an avalanche-type manner and a prognosis of the fracture process is unjustified without taking account of dynamic effects.

Under the assumption that $K'_{1,1} > 0$ and $K'_{2,1} = 0$, in the case of an isotropic material (5.9), formulae (6.2) and (6.3) take the form

$$a = \frac{32}{3}(2\pi)^{-1/2}\frac{1-\nu}{\mu}(K_{2,1}^0)^2\frac{K_{1,2}^0}{K_{1,1}^0} + O(\varepsilon^3)$$

$$b = -\tau\frac{1-\nu}{\mu}((K_{1,1}^0)^2 + 2(K_{2,1}^0)^2)\frac{K'_{1,1}}{K_{1,1}^0} + O(\varepsilon^3)$$

$$h(\tau) = \frac{\pi}{32}\left(1+\frac{1}{2}\left(\frac{K_{1,1}^0}{K_{2,1}^0}\right)^2\right)^2\left(\frac{K'_{1,1}}{K_{1,2}^0}\right)^2\tau^2 + O(\tau^3) \tag{6.4}$$

In representations (3.1) and (3.7) of the energy increments $\Delta\Pi$ and ΔU, we collect the factors accompanying h^2 and, applying formulae (3.8), (3.9) and (4.8), (4.14), (4.18), (4.21) for the elements of the energy release matrix, we find

$$\begin{aligned} T_4^0 = {} & \varepsilon^2\left(\frac{9}{8}H_3^2 + 2H_2H_4\right)(\gamma_0+\gamma_0'') + \varepsilon^2 H_4 M_{22}K_{1,1}^0\overline{K}_{2,1}^0 + \\ & + \frac{3}{4}(2\pi)^{1/2}\varepsilon^2 H_3 B\left(\frac{3}{4}H_2K_{1,1}^0 - \overline{K}_{2,1}^0\right)K_{1,2}^0 - \frac{1}{2}\varepsilon^2\mathcal{M}_{1,2}^{1,2(2)}(0)(K_{1,2}^0)^2 - \frac{1}{2}\sum_j\sum_p K_{j,1}^0(M_{jj}\mathfrak{L}_{p,1}^{j,1}M_{pp} - \\ & - 2\varepsilon H_2M_{22}\mathfrak{L}_{p,1}^{3-j,1}M_{pp} + \varepsilon^2H_2^2M_{22}^2\mathfrak{L}_{3-p,1}^{3-j,1})K_{p,1}^0 + K_{1,1}^0M_{11}K_{1,3}^0 + \ldots \end{aligned} \tag{6.5}$$

Here, $\mathcal{M}^{1,2(2)}_{1,2}(h)$ is the third term of expansion (3.8) of the element $\mathcal{M}^{1,2}_{1,2}(h)$ of the energy release matrix. It had not been found explicitly in Section 4 but the value of $\mathcal{M}^{1,2(2)}_{1,2}(0)$, calculated when $h=0$ and appearing in relation (6.5), depends solely dependent on the coefficient $H_2(\varepsilon)$ of series (1.4).

By virtue of formula (5.13) for H_2, the terms containing H_4 cancel out (the terms with H_3 in expression (6.1) disappeared in a similar manner). According to the Griffith criterion, the function (6.5) must reach a maximum value with respect to the unknown H_3. Consequently, the quantity εH_3, which, according to relation (1.4), determines the form of the deflection of the branch of the crack, has the form

$$\varepsilon H_3 = \frac{1}{3}(2\pi)^{1/2}(\gamma_0+\gamma_0'')^{-1}B\left(1+\frac{3}{2}\left(1+\frac{\gamma_0''}{\gamma_0}\right)^{-1}\frac{M_{22}}{M_{11}}\right)K^0_{2,1}K^0_{1,2}+O(\varepsilon^2) \tag{6.6}$$

For an isotropic material, the right-hand side of relation (6.6) is equal to

$$\frac{5}{12}(2\pi)^{1/2}\gamma_0^{-1}\frac{1-\nu}{\mu}K^0_{2,1}K^0_{1,2}+O(\varepsilon^2) = \frac{5}{3}(2\pi)^{1/2}\frac{K^0_{2,1}K^0_{1,2}}{(K^0_{1,1})^2}+O(\varepsilon^2)$$

In the case of stable growth of the crack, taking the quadratic term $h^2T_4(\tau)$ in the functional (5.12) into account enables us to refine expression (6.3) for the length of the branch

$$h(\tau) = \frac{4}{9}\left(\frac{T^1_2}{T^0_3}\right)^2\left(\tau^2-2\tau^3\left(\frac{T^1_3}{T^0_3}+\frac{4}{9}\frac{T^0_4}{T^0_3}\left|\frac{T^1_2}{T^0_3}\right|\right)+O(\tau^4)\right)$$

However, this formula is implicit when $K^0_{1,2}\neq 0$ since the coefficient $\mathcal{M}^{1,2(2)}_{1,2}(0)$ in relation (6.5) has not been determined.

7. Scenarios for the crack growth without longitudinal loading

Suppose $K^0_{1,2}=0$ in addition to requirements (5.8). Then, according to formulae (5.13) and (6.6), expression (6.1) becomes negligibly small and the trinomial (5.12) is replaced, with a permissible error, by the sum

$$h\tau T^1_2+h^2T^0_4 \tag{7.1}$$

On retaining quantities $O(1)$ compared with the small parameter ε on the right-hand sides of relations (6.5) and (5.11), we find the coefficients of the binomial (7.1)

$$T^0_4 = -\frac{1}{2}M_{11}\mathcal{L}^{1,1}_{1,1}(K^0_{1,1})^2+M_{11}K^0_{1,1}K^0_{1,3}+O(\varepsilon),\quad T^1_2 = M_{11}K^0_{1,1}K'_{1,1}+O(\varepsilon) \tag{7.2}$$

A graph of the function

$$\mathbb{R}_+\ni h\mapsto ah^2+bh$$

is shown in Fig. 1.

Putting $a=T^0_4$ and $b=T^1_2$, we see that one of the following possibilities is occurs realized under the above-mentioned conditions.

1°. If $T^0_4>0$ and $T^1_2>0$ (in particular, the SIF $K^0_{1,1}+\tau K'_{1,1}$ decreases), crack stops, that is, a minimum of the functional (7.1) is reached when $h=0$.

2°. If $T^0_4>0$ and $T^1_2<0$ (the SIF $K^0_{1,1}+\tau K'_{1,1}$ increases), the crack develops quasistatically and stably, and the position of its tip at an instant τ is found from the formula

$$h(\tau) = -\frac{1}{2}\frac{T^1_2}{T^0_4}\tau+O(\tau^2) \tag{7.3}$$

3°. If $T^0_4<0$ and $T^1_2>0$, the crack is unstable and, after passing into the position (7.3), corresponding to a maximum of the functional (7.1), an avalanche-type growth of the crack occurs.

4°. If $T^0_4<0$ and $T^1_2<0$, the fracture process is of the avalanche-type from the very beginning and a quasistatic model is unacceptable.

8. Discussion

According to formula (5.16), the angle of deviation of the branch $\Upsilon(\tau)$ from the Ox_1 axis is opposite to the sign of the SIF $K^0_{2,1}$. In the case of a stretching longitudinal stress $K^0_{1,2}>0$, the quantity (6.6) has the same sign as the SIF $K^0_{2,1}$, that is, distortion of the branch $\Upsilon(\tau)$ brings its tip closer to the axis of the initial crack L. In the case of a compressive longitudinal stress $K^0_{1,2}<0$, the branch deviates in the direction of the Ox_1 axis. However, it has already been established in this case that the fracture process acquires a dynamic character and a prognosis of the form of the crack on the basis of a quasistatic model is hardly possible.

As would be expected, the critical load, found from relation (5.15), and the parameters (5.16) and (6.6) of the crack trajectory are calculated only using the SIFs $K^0_{1,1}$, $K^0_{2,1}$ and the longitudinal stress $K^0_{1,2}$, that is, using the characteristics of the stress-strain state at the instant $\tau = 0$. However, the length of the branch $\Upsilon(\tau)$ and, in particular, the start of the crack are determined from to formulae (6.3) or (7.3) taking account of the SIFs $K'_{1,1}$ and $K'_{2,1}$ which are caused by the evolution of the load, that is, by the term $\tau g'$ in representation (1.3). Note that, when $\tau > 0$, the SIFs $K^0_{j,1} + \tau K'_{j,1}$ are fictitious: they correspond to a load $g(\tau)$ which has changed but for the initial position of the crack.

The properties of the fracture process under conditions of longitudinal stress $K^0_{1,2} \neq 0$ have been established in Section 5 on the basis of the local characteristics $K^0_{j,1}$, $K'_{j,1}$ and $K^0_{1,2}$ of the stress state at the mouth of the crack. This is unexpected since, both in the case of the rectilinear propagation of a crack[13,19] as well as in the case when $K^0_{1,2} = 0$, the state of the crack, which is a quiescent state, a stable quasistatic state or one of dynamic growth, depends on the element $\mathcal{L}^{1,1}_{1,1}$ of the matrix $\mathcal{L}$ (see formulae (7.2) and (7.3)). The coefficient $\mathcal{L}^{1,1}_{1,1}$ of the expansion of the weighting function $\zeta^{1,1}$ is a global characteristic of the body Ω and the crack L. Hence, when there is no longitudinal stress in the bodies $\Omega^1\backslash L^1$ and $\Omega^2\backslash L^2$ but there is the same stress-strain state near their tips, the fracture process can occur in a different was. For example, in the case of a small length, a boundary crack L usually grows in a stable manner but, on approaching the opposite boundary of the body, it always grows in an avalanche-type manner. Meanwhile, the external loads can be selected such that the SIFs $K^0_{j,1}$, $K'_{j,1}$ and $K^0_{1,2}$ are the same for both positions of the crack tip.

We now point out a further special feature of the crack growth under a longitudinal stress $K^0_{1,2} > 0$. It starts with zero velocity $\partial\tau h(0)$ but with a large acceleration $\partial^2_\tau h(0)$, since the factor accompanying τ^2 in expressions (6.3) and (6.4) contains the small SIF $K^0_{2,1}$ of the second mode in the denominator. This fact reduces the permissible upper boundary of the change in the time-like parameter τ, that is, in order to maintain the accuracy of the asymptotic formulae, it is necessary that the product $\varepsilon^{-2}\tau$ should remain small. However, if $K'_{1,1} = O(\varepsilon^2)$ and $K'_{2,1} = O(\varepsilon)$, then not only the denominator (5.8) turns out to be small but, also, the numerator (5.11) in relation (6.3) and, consequently, additional constraints on the magnitude of τ are not necessary.

The factor $\kappa = (1 + \gamma''_0/\gamma_0)^{-1}$ in formulae (5.15), (5.16) and (6.2) reflects the anisotropy of the strength characteristics of the material. Since it is assumed that $\gamma'_0 = 0$, the inequality $\kappa < 1$ implies that the surface energy density has a strict minimum at the point $\theta = 0$. At the same time, according to formulae (5.16) and (5.15), the angle of deviation of the crack decreases and the critical load increases compared with the isotropic case. If, however, $\kappa > 1$ and the density γ has a strict maximum when $\theta = 0$, then the opposite pattern is observed. Moreover, when $\kappa \gg 1$, the rapid decrease in $\gamma(\theta)$ accompanying the growth in $|\theta|$ leads to a significant deviation of the crack form the initial direction and asymptotic methods cannot be used to solve the problem.

For any scenario, the length of the branch does not depend solely on the SIFs $K^0_{j,1}$ and $K'_{j,1}$ but, also, on the coefficients accompanying the singularities $o(r^{1/2})$, that is, on the longitudinal stress $K^0_{1,2}$ and the lowest SIFs $K^0_{1,3}$ in Section 7.

Acknowledgement

The research was financed by the Russian Foundation for Basic Research (03-01-00835).

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Translated by E.L.S.